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HEAT TRANSFER AND SKIN FRICTION OF A PHASE-CHANGING INTERFACE OF GAS–LIQUID LAMINAR FLOWS

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Abstract—Gas and liquid laminar flows having a phase-changing (evaporation or condensation) interface at their common boundary are investigated numerically under the conditions of constant properties and of flat-surface boundary layers of zero-pressure gradient. The increase of the normal velocity at the interface associated with phase-changing modifies the velocity and temperature profiles so as to reduce the coefficients of skin-friction and heat-transfer at the interface. With an approximation for the velocity profile, these coefficients are analytically presented as functions of the parameter of the phase-change, that is, the normal and parallel velocities and the temperature or the vapor concentration at the interface.

NOMENCLATURE

f, non-dimensional stream function;
g, non-dimensional temperature;
H, $= 2c_2(T_{1\infty} - T_{2\infty})/L$;
L, latent heat of vaporization;
p, pressure;
P, Prandtl number (ν/κ);
S, Schmidt number (ν/ϵ);
T, temperature;
u, v, x and y components of velocity;
w, mass-fraction of vapor;
x, y, co-ordinates.

Greek symbols

Δ, $= f_0 - (f'_0)^2/(2f''_0)$;
ϵ, diffusion coefficient of vapor;
η, non-dimensional co-ordinate of y;
κ, thermal diffusivity;
Λ, $= \rho_2/\rho_1/(\nu_2/\nu_1)$;
λ, heat conductivity;
ν, kinematic viscosity;
ρ, density;
ψ, stream function.

Subscripts

0, interface of gas and liquid;
1, gas side;
2, liquid side;
∞, at infinity.

INTRODUCTION

THE SOLUTIONS of simultaneous heat and mass transfer at the interface between gas and liquid flows where evaporation or condensation is in progress indicate basic behavior patterns in engineering problems of the heat transfer of two-phase flow, transpiration or mass-transfer cooling, drying process etc. or in geophysical problems of water evaporation at the surface of ocean or land. Concerning transpiration cooling, mist cooling, ablation etc. much work has been published because they offered great promise for the maintenance of tolerable surface temperature on high speed aircraft, turbine blades or rocket-motor nozzle. A preliminary and fundamental approach to such problems is the examination of the boundary-layer flows under the condition that the quantities (velocity, temperature, etc.) at the interface are given explicitly. For two-layer flows of gas and liquid which have an appreciable velocity at the interface to force the liquid into motion, or a large gradient of temperature in the liquid layer, the boundary conditions at the interface should however be provided implicitly by the continuity relationship of mass, momentum and energy flows through the surface between two layers.

This study is concerned with the theoretical prediction of the heat and mass transfer and the skin friction for such phase-changing (evaporating or condensing) interfaces between laminar boundary-layer flows of gas and liquid. The fields of velocity, temperature and concentration are interacted each other at the interface so that their solutions, if they exist, should be given as eigen functions corresponding to the boundary conditions at the interface. Firstly, a mathematical description of the problem is presented and solved numerically. Then, in order to make it perspective, their analytical solutions are examined approximately.

LAMINAR FLOWS OF GAS AND LIQUID HAVING A PHASE-CHANGING INTERFACE

Over a liquid layer, gaseous fluid containing vapor of the liquid flows steadily and sufficiently slowly to form a laminar boundary layer. At the boundary of two layers, the interface of gas and liquid layers, evaporation or condensation takes place corresponding to the state of the vapor relative to its saturation. Let x denote the coordinate parallel to the interface, y the co-ordinate normal to it, and (u, v) the corresponding velocity components. Properties of gas and liquid are denoted by the subscripts 1 and 2 respectively. In addition, the free-stream of gas just outside the boundary layer has the velocity $u_{1\infty}$ and the temperature $T_{1\infty}$. The velocity and temperature of the liquid far from the interface are $u_{2\infty}$ and $T_{2\infty}$, respectively. Without any loss of generality, we can assume a zero velocity of the liquid at infinity, so that in the following we are concerned with the case

$$u_{1\infty} = u_{\infty}, \qquad u_{2\infty} = 0.$$

The mass-fraction (or concentration) of the vapor in the gas side is denoted by w which is w_{∞} at the outer edge of the boundary layer.

For the case of two-dimensional steady laminar flow with negligible dissipation of an incompressible fluid with constant properties, the boundary-layer equations of momentum, energy and diffusion in each region of gas and liquid can be written as follows in the vector form;

$$u\frac{\partial}{\partial x}\begin{pmatrix} u \\ T \\ w \end{pmatrix} + v\frac{\partial}{\partial y}\begin{pmatrix} u \\ T \\ w \end{pmatrix} = (\nu, \kappa, \epsilon)\frac{\partial^2}{\partial y^2}\begin{pmatrix} u \\ T \\ w \end{pmatrix} \tag{1}$$

where ν is the kinematic viscosity, κ the thermal diffusivity and ϵ the diffusion coefficient of the vapor, being assumed to be constant throughout the layers. The continuity of mass is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2}$$

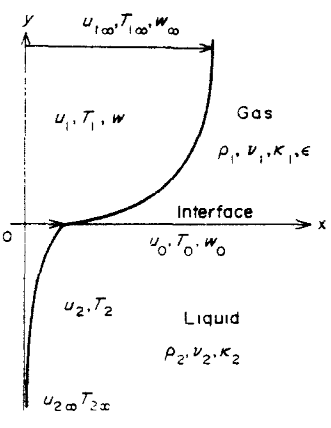

FIG. 1. Phase-changing interface of gas and liquid boundary-layer flows.

Let us consider the boundary conditions at the interface of gas and liquid layer. The no-slip condition of the temperature and velocity at the interface requires that

$$T_{10} = T_{20} \tag{3}$$

$$u_{10} = u_{20}. \tag{4}$$

The continuity of mass gives

$$(\rho_1 v_1)_0 = (\rho_2 v_2)_0 \tag{5}$$

$$(\rho_1 v_1)_0(1 - w_0) = -\rho_1 \epsilon \left(\frac{\partial w}{\partial y}\right)_0. \tag{6}$$

The shearing force balance in the flow direction is

$$\rho_1 \nu_1 \left(\frac{\partial u_1}{\partial y}\right)_0 = \rho_2 \nu_2 \left(\frac{\partial u_2}{\partial y}\right)_0 \tag{7}$$

and the energy balance that the heat transferred to the interface is entirely exhausted for the phase change is

$$L(\rho_1 v_1)_0 = \lambda_1 \left(\frac{\partial T_1}{\partial y}\right)_0 - \lambda_2 \left(\frac{\partial T_2}{\partial y}\right)_0 \tag{8}$$

where L is the latent heat of evaporation and λ the thermal conductivity. The vapor of the liquid has the pressure of saturation at the temperature of the interface T_0, which holds Clausius–Clapeyron's relation

$$\frac{\mathrm{d}p_s}{\mathrm{d}T_0} \approx \frac{L}{R_v}\frac{p_s}{T_0^2}.$$

This yields the mass-fraction, that is, the partial pressure of the vapor at the interface;

$$w_0 = \frac{p_s}{p_0} = \exp\left[\frac{L}{R_v}\left(\frac{1}{T_b} - \frac{1}{T_0}\right)\right] \tag{9}$$

where R_v is the gas constant of the vapor, T_b the boiling point at the ambient pressure p_∞.

The foregoing equations and boundary conditions can be transformed to a more tractable form by introducing the stream function

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

which satisfies the continuity equation of mass (2), as well as the following nondimensionalized co-ordinate η and the nondimensionalized stream function f;

$$\eta_1 = \sqrt{\left(\frac{u_\infty}{\nu_1 x}\right)}\cdot y, \qquad \eta_2 = \sqrt{\left(\frac{u_\infty}{\nu_2 x}\right)}\cdot y$$

$$\psi_1 = \sqrt{(u_\infty \nu_1 x)}\,.f_1(\eta_1),$$

$$\psi_2 = \sqrt{(u_\infty \nu_2 x)}\,.f_2(\eta_2).$$

With these substitutions and the assumption of fully developed flows that f, T and w are functions of η only, equations (1) are reduced to

$$2f''' + ff'' = 0 \tag{10}$$

$$2T'' + PfT' = 0 \tag{11}$$

$$2w'' + Sfw' = 0 \tag{12}$$

where the subscripts 1 and 2 are omitted and a prime denotes an appropriate derivation with respect to η. P is the Prandtl number (ν/κ) and S the Schmidt number (ν/ϵ). In addition, defining of nondimensionalized temperature as

$$g_1 = \frac{T - T_{1\infty}}{T_{2\infty} - T_{1\infty}}, \qquad g_2 = \frac{T - T_{2\infty}}{T_{1\infty} - T_{2\infty}}$$

reduces equation (11) to

$$2g'' + Pfg' = 0. \tag{11'}$$

The above substitutions of f and g rewrite the boundary conditions (3)–(8) as follows. The continuity relations of temperature and velocity, (3) and (4) are

$$g_{10} + g_{20} = 1 \tag{13}$$

$$f'_{10} = f'_{20} \tag{14}$$

and the mass continuity, equations (5) and (6) are

$$f_{10} = \Lambda f_{20} \tag{15}$$

$$Sf_{10}(1 - w_0) = 2w'_0 \tag{16}$$

where

$$\Lambda = \frac{\rho_2}{\rho_1}\sqrt{\left(\frac{\nu_2}{\nu_1}\right)}.$$

The shearing force balance (7) and the energy continuity (8) become

$$f''_{10} = \Lambda f''_{20} \tag{17}$$

$$f_{10} = m_1 g'_{10} + m_2 g'_{20} \tag{18}$$

where

$$m_1 = H\frac{c_1}{c_2}\frac{\kappa_1}{\nu_1}, \quad m_2 = \Lambda H\frac{\kappa_2}{\nu_2},$$

$$H = \frac{2c_2(T_{1\infty} - T_{2\infty})}{L},$$

and c_1 and c_2 are the specific heats at constant pressure of gas and liquid, respectively.

Denoting the first derivative of f as F:

$$F_1 = f'_1, \quad F_2 = f'_2 \tag{19}$$

that is,

$$f_1 = f_{10} + \int_0^{\eta_1} F_1 \, d\eta_1, \quad f_2 = f_{20} - \int_{\eta_2}^{0} F_2 \, d\eta_2 \tag{19'}$$

we obtain from equation (10)

$$2F'' + fF' = 0$$

which yields

$$F_1 = F_{10} + F'_{10}\gamma_1(1, \eta_1),$$
$$F_2 = F_{20} - F'_{20}\gamma_2(1, \eta_2) \tag{20}$$

where

$$\gamma_1(P, \eta) = \int_0^{\eta} \exp\left(-\frac{P}{2}\int_0^{\eta} f_1 \, d\eta\right) d\eta,$$

$$\gamma_2(P, \eta) = \int_{\eta}^{0} \exp\left(\frac{P}{2}\int_{\eta}^{0} f_2 \, d\eta\right) d\eta.$$

The boundary conditions that at $\eta_1 = \infty$ and at $\eta_2 = -\infty$

$$F_{1\infty} = 1, \quad F_{2\infty} = 0$$

and that at the interface $\eta_{1,2} = 0$

$$F_{10} = F_{20}, \quad F'_{10} = \Lambda F'_{20}$$

yield the relation

$$1 = F_{10} + F'_{10}\gamma_1(1, \infty),$$
$$0 = \Lambda F_{10} - F'_{10}\gamma_2(1, -\infty)$$

that is,

$$\left.\begin{aligned} F'_{10} = f''_{10} &= \frac{\Lambda}{\Lambda\gamma_1(1, \infty) + \gamma_2(1, -\infty)} \\ F_{10} = f'_{10} &= \frac{\gamma_2(1, -\infty)}{\Lambda\gamma_1(1, \infty) + \gamma_2(1, -\infty)}. \end{aligned}\right\} \tag{21}$$

On the other hand, from equations (11′) and (12) we obtain the temperature and concentration fields that

$$\left.\begin{aligned} g_1 &= g_{10}\left(1 - \frac{\gamma_1(P_1, \eta_1)}{\gamma_1(P_1, \infty)}\right) \\ g_2 &= g_{20}\left(1 + \frac{\gamma_2(P_2, \eta_2)}{\gamma_2(P_2, -\infty)}\right) \\ w - w_\infty &= (w_0 - w_\infty)\left(1 - \frac{\gamma_1(S, \eta_1)}{\gamma_1(S, \infty)}\right). \end{aligned}\right\} \tag{22}$$

Substitution of equations (22) into the boundary conditions (16) and (18) gives the relationships between the normal velocity, the temperature and the concentration at the interface:

$$\frac{S}{2} f_{10} = \frac{w_\infty - w_0}{1 - w_0}\frac{1}{\gamma_1(S, \infty)} \tag{23}$$

$$f_{10} = -\frac{m_1 g_{10}}{\gamma_1(P_1, \infty)} + \frac{m_2 g_{20}}{\gamma_2(P_2, -\infty)}. \tag{24}$$

The temperature and its gradient at the interface, g_{10} and g'_{10} are thus obtained from equations (13) and (22):

$$g_{10} = \frac{-f_{10} + \dfrac{m_2}{\gamma_2(P_2, -\infty)}}{\dfrac{m_1}{\gamma_1(P_1, \infty)} + \dfrac{m_2}{\gamma_2(P_2, -\infty)}} \tag{25}$$

$$g'_{10} = -\frac{1}{\gamma_1(P_1, \infty)}\,\frac{-f_{10} + \dfrac{m_2}{\gamma_2(P_2, -\infty)}}{\dfrac{m_1}{\gamma_1(P_1, \infty)} + \dfrac{m_2}{\gamma_2(P_2, -\infty)}} \tag{26}$$

With equation (9), the relationship between w_0 and g_0, the normal velocities f_{10} and $f_{20}(=\Lambda^{-1}f_{10})$ can be obtained from equations (23) and (25), which are the implicit functions of f_1 and f_2; The velocity fields f_1 and f_2 are the solution of integral equations (19′) which involves f_{10} and f_{20} through equations (20) and (21). Thus, the iterative method may readily present their numerical solutions. The numerical results obtained in such a way are shown in Figs. 2–8. The used fluids are air for the gas and water, benzene and methyl alcohol for the liquid. The physical properties of gas and liquid are evaluated at $T_{1\infty}$ and $T_{2\infty}$, respectively.

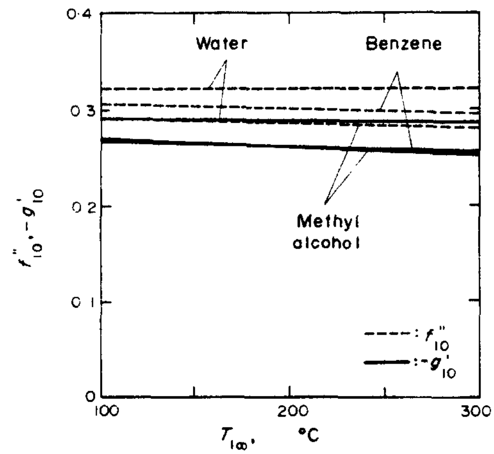

FIG. 2. Effect of air temperature $T_{1\infty}$ on friction and heat-transfer coefficients ($T_{2\infty} = 20°C$, $w_\infty = 0$).

The skin-friction coefficient, c_f, and the heat-transfer coefficient, c_h, for the gas side are defined as

$$c_f = \frac{\rho_1 \nu_1 (\partial u/\partial y)_{10}}{\rho_1 (u_{1\infty} - u_{2\infty})^2} = f''_{10}\sqrt{\left(\frac{\nu_1}{u_\infty x}\right)}$$

$$c_h = \frac{\lambda_1 (\partial T/\partial y)_{10}}{\lambda_1 (T_{1\infty} - T_{2\infty})/x} = -g'_{10}\sqrt{\left(\frac{\nu_1}{u_\infty x}\right)}$$

so that f''_{10} and $-g'_{10}$ mean the dimensionless coefficients, being 0·332 for the Blasius profile.

Figures 2 and 3 indicate the effect of air and liquid temperature, $T_{1\infty}$ and $T_{2\infty}$ on the friction coefficient f''_{10} and the heat-transfer coefficient $-g'_{10}$. These figures show that the coefficients are greatly dominated by the liquid temperature, although they depend slightly upon the air temperature. The dimensionless profiles of velocity, temperature and concentration across the boundary layer for the air–water case are shown in Fig. 4 for various liquid temperatures. Figures 5 and 6 show the effect of the vapor concentration outside the boundary layer upon

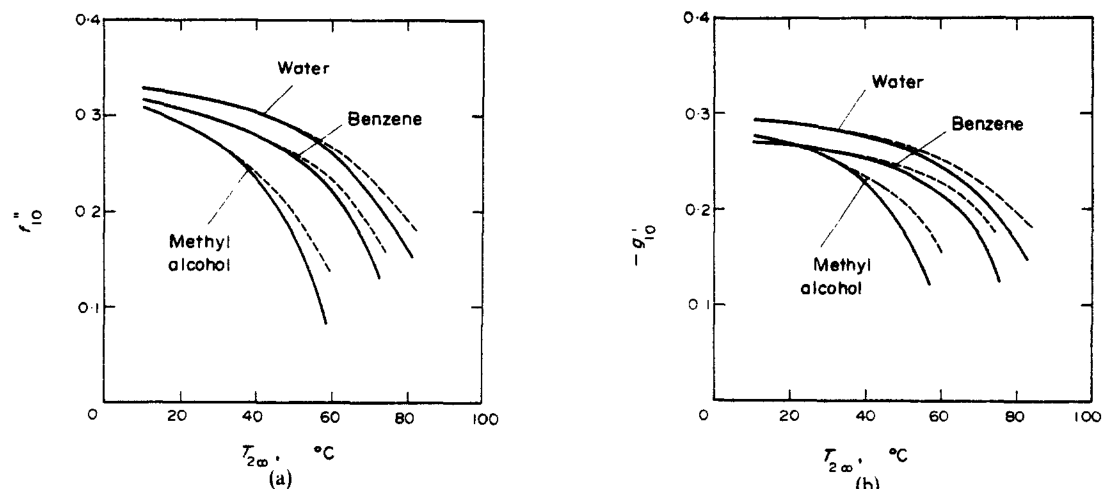

FIG. 3. Effect of liquid temperature $T_{2\infty}$ on friction (a) and heat-transfer (b) coefficients ($T_{1\infty} = 100°C$, $w_\infty = 0$).

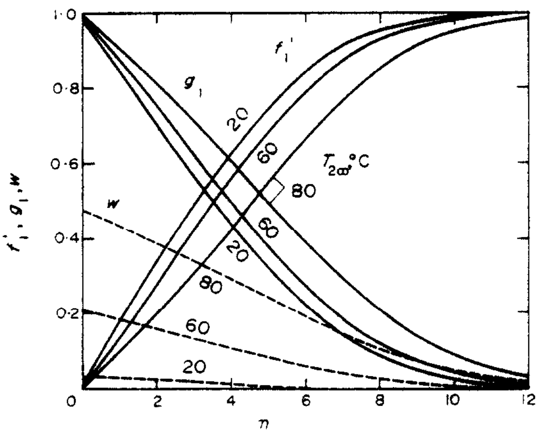

FIG. 4. Profiles of u-velocity, temperature and vapor concentration at various liquid temperature $T_{2\infty}$ (water, $T_{1\infty} = 100°C$, $w_\infty = 0$).

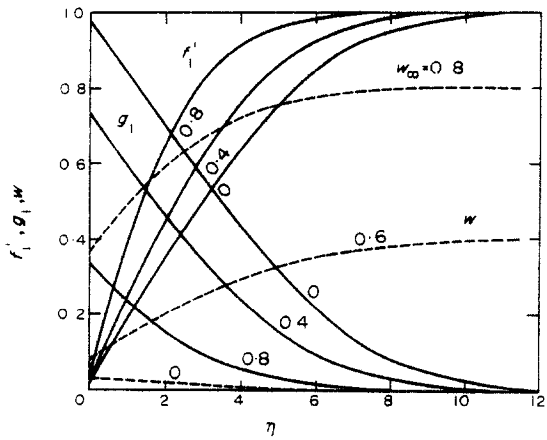

FIG. 6. Profiles of u-velocity, temperature and vapor concentration for vapor concentration w_∞ (water, $T_{1\infty} = 100°C$, $T_{2\infty} = 20°C$).

the coefficients and Fig. 7 is the effect of the latent heat of vaporization.

At first sight to equation (26), it is noted that, when $T_{1\infty} > T_{2\infty}$, that is, m_1 and m_2 are positive, the phase-change ($f_{10} \neq 0$) should make the heat-transfer coefficient $|g'_{10}|$ increase or decrease, corresponding to evaporation ($f_{10} < 0$) or condensation ($f_{10} > 0$), respectively. The numerical results do not necessarily show such a behavior. This means the importance of the influence of the phase-change at the interface upon the velocity distributions f_1 and f_2 which are involved implicitly in γ_1 and γ_2.

HEAT TRANSFER AND SKIN FRICTION

To make the role of the phase-change at the interface more perspective, let us treat the problem analytically in an approximate method.

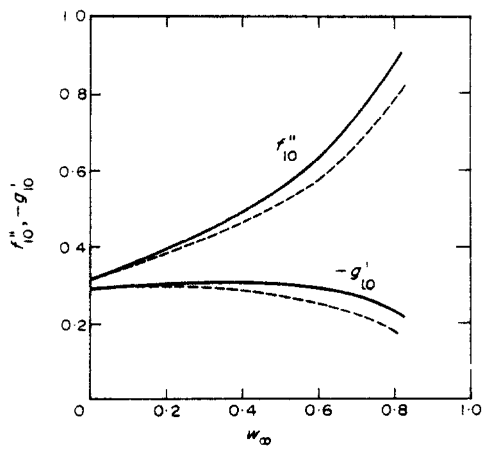

FIG. 5. Effect of vapor concentration w_∞ on friction and heat-transfer coefficients (water $T_{1\infty} = 100°C$, $T_{2\infty} = 20°C$).

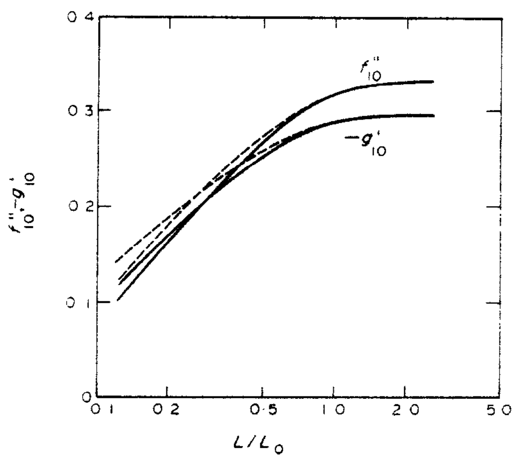

FIG. 7. Effect of latent heat of vaporization L on friction and heat-transfer coefficients (water, $T_{1\infty} = 100°C$, $T_{2\infty} = 20°C$, $w_\infty = 0$, $L_0 = 539$ cal/g).

Equations (1) or (10)–(12) are independent of each other so that the fields of velocity, temperature and concentration should not be interacted through these basic equations. Their interactions result from the boundary conditions at the gas–liquid interface. The change in the phase of the fluid at the interface, controlled by the temperature field, causes a mass flow normal to the surface to modify the velocity and temperature distributions across the layers. Such an interaction of the fields through the boundary conditions at the interface can be represented explicitly in the basic equations with the use of an intrinsic co-ordinate of the stream function.

Denote z the intrinsic co-ordinate with its origin $\bar{\psi}_0(x)$ which is determined later;

$$z = \psi - \bar{\psi}_0$$

and transform the (x, y) co-ordinates into (x, z) with the relation

$$\frac{\partial}{\partial y} = u\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial x} - \left(v + \frac{\partial \bar{\psi}_0}{\partial x}\right)\frac{\partial}{\partial z}$$

then, equation (1) can be reduced to

$$\frac{\partial}{\partial x}\begin{pmatrix} u \\ T \\ w \end{pmatrix} - \frac{\partial \bar{\psi}_0}{\partial x}\frac{\partial}{\partial z}\begin{pmatrix} u \\ T \\ w \end{pmatrix} = (\nu, \kappa, \epsilon)\frac{\partial}{\partial z}\left\{u\frac{\partial}{\partial z}\begin{pmatrix} u \\ T \\ w \end{pmatrix}\right\} \quad (27)$$

of which the second term in the left hand side expresses the non-linear interaction at the interface. In the neighbourhood of the interface, we can approximate the stream function as

$$\psi - \psi_0 \approx u_0 y + \tfrac{1}{2}u_{y0}y^2$$

where ψ_0 is the value of the stream function at the interface and $u_y = \partial u/\partial y$, and the streamwise velocity as

$$u \approx u_0 + u_{y0}y$$

so that

$$\psi - \psi_0 = \frac{1}{2u_{y0}}(u^2 - u_0^2).$$

Here, we take the value of $\bar{\psi}_0$ as

$$\bar{\psi}_0 = \psi_0 - \frac{u_0^2}{2u_{y0}} \quad (28)$$

then the velocity can be expressed in the form

$$u = \sqrt{(2u_{y0}z)} \quad (29)$$

and equations (27) are reduced to

$$(\nu, \kappa, \epsilon)\frac{\partial}{\partial z}\left\{\sqrt{(2u_{y0}z)}\frac{\partial}{\partial z}\begin{pmatrix} u \\ T \\ w \end{pmatrix}\right\} - \frac{\partial}{\partial x}\begin{pmatrix} u \\ T \\ w \end{pmatrix} = -\frac{\partial \bar{\psi}_0}{\partial x}\frac{\partial}{\partial z}\begin{pmatrix} u \\ T \\ w \end{pmatrix}. \quad (30)$$

Firstly, let us consider the velocity field. Using the transformation of

$$\xi = \int_0^x \nu\sqrt{(2u_{y0})}\,\mathrm{d}x$$

and the Heaviside operator s corresponding to $\partial/\partial\xi$, we can transform equation (30) to

$$\frac{\partial}{\partial z}\left(\sqrt{(z)}\frac{\partial[u - u_\infty]}{\partial z}\right) - s[u - u_\infty] = -\left[\frac{1}{\nu\sqrt{(2u_{y0})}}\frac{\partial \bar{\psi}_0}{\partial x}\frac{\partial(u - u_\infty)}{\partial z}\right]$$

where the Heaviside operated value is denoted by [ ]. By setting

$$\zeta = \sqrt{z}$$

the above equation can be transformed to a more amenable form

$$\frac{\partial^2[u - u_\infty]}{\partial \zeta^2} - 4\zeta s[u - u_\infty] = -[Q_u] \quad (31)$$

where

$$Q_u = \frac{2}{\nu\sqrt{(2u_{y0})}}\frac{\partial \bar{\psi}_0}{\partial x}\frac{\partial(u - u_\infty)}{\partial \zeta} \approx \frac{2}{\nu}\frac{\partial \bar{\psi}_0}{\partial x}.$$

The solution of equation (31) is given by

$$[u - u_\infty] = \sqrt{(\zeta)}\{I_{\frac{1}{3}}(Z)[A_1(\zeta) + C_1] + I_{-\frac{1}{3}}(Z)[A_2(\zeta) + C_2]\} \quad (32)$$

where I_n is the modified Bessel function of the first kind of order n and

$$Z = \tfrac{4}{3}\sqrt{(s)}\zeta^{\frac{3}{2}}$$

$$A_1(\zeta) = -\frac{2\pi}{3\sqrt{3}}\int_0^{\zeta}\sqrt{(\zeta)}I_{-\frac{1}{3}}(Z)\,.\,[Q_u]\,\mathrm{d}\zeta$$

$$A_2(\zeta) = \frac{2\pi}{3\sqrt{3}}\int_0^{\zeta}\sqrt{\zeta}I_{\frac{1}{3}}(Z)\,.\,[Q_u]\,\mathrm{d}\zeta.$$

C_1 and C_2 are the constants which are determined by the boundary conditions at $z = 0$

$$[u - u_\infty]_{\zeta\to 0} = [\sqrt{(2u_{y0})}\zeta - u_\infty]_{\zeta\to 0}$$

which gives

$$C_1 = \Gamma(\tfrac{4}{3})\,.\,(\tfrac{2}{3}\sqrt{s})^{-\frac{1}{3}}[\sqrt{(2u_{y0})}],$$
$$C_2 = \Gamma(\tfrac{2}{3})\,.\,(\tfrac{2}{3}\sqrt{s})^{\frac{1}{3}}[-u_\infty]$$

where Γ is the gamma function. The condition $[u - u_\infty] \to 0$ as $z \to \infty$ leads to

$$\Gamma(\tfrac{4}{3})\,.\,(\tfrac{2}{3}\sqrt{s})^{-\frac{1}{3}}[\sqrt{(2u_{y0})}] + \Gamma(\tfrac{2}{3})\,.\,(\tfrac{2}{3}\sqrt{s})^{\frac{1}{3}}[-u_\infty] = \frac{2\pi}{3\sqrt{3}}[Q_u]\int_0^\infty \sqrt{(\zeta)}\{I_{-\frac{1}{3}}(Z) - I_{\frac{1}{3}}(Z)\}\,\mathrm{d}\zeta. \quad (33)$$

Since the integral of the right hand side is $(2\sqrt{s})^{-1}$, equation (33) becomes

$$\Gamma(\tfrac{4}{3})\,.\,(\tfrac{2}{3})^{-\frac{1}{3}}[\sqrt{(2u_{y0})}] = \Gamma(\tfrac{2}{3})\,.\,(\tfrac{2}{3})^{\frac{1}{3}}s^{\frac{1}{3}}[u_\infty] + \frac{\pi}{3\sqrt{3}}s^{-\frac{1}{3}}[Q_u]. \quad (34)$$

The inversion transform of this equation yields

$$\Gamma(\tfrac{1}{3})\,.\,(\tfrac{2}{3})^{-\frac{1}{3}}\sqrt{(2u_{y0})} = (\tfrac{2}{3})^{\frac{1}{3}}u_\infty\{\int_0^x \nu\sqrt{(2u_{y0})}\,\mathrm{d}x\}^{-\frac{1}{3}} + \frac{\pi}{3\sqrt{3}}\frac{1}{\Gamma(\frac{1}{3})}\int_0^x\{\int_{x'}^x \nu\sqrt{(2u_{y0})}\,\mathrm{d}x''\}^{-\frac{2}{3}} \times \nu\sqrt{(2u_{y0})}\,.\,Q_u\,\mathrm{d}x'. \quad (35)$$

Using the expression (f, η) for (u, y), we obtain

$$u_0 = u_\infty f'_0, \quad v_0 = -\frac{1}{2}\sqrt{\left(\frac{u_\infty \nu}{x}\right)}f_0,$$

$$u_{y0} = u_\infty\sqrt{\left(\frac{u_\infty}{\nu x}\right)}f''_0, \quad Q_u = \sqrt{\left(\frac{u_\infty}{\nu x}\right)}\Delta$$

where Δ is defined as

$$\Delta = f_0 - \frac{(f'_0)^2}{2f''_0} \quad (36)$$

which represents the amount of the phase-change and the fluid motion at the interface. Setting these relations into equation (35) yields

$$f''_0 = A(f''_0)^{\frac{1}{3}}\{1 + B(f''_0)^{\frac{1}{3}}\Delta\} \quad (37)$$

where

$$A = (\tfrac{3}{2})^{\frac{2}{3}}/\Gamma(\tfrac{1}{3}) = 0{\cdot}489,$$

$$B = \frac{\pi}{\sqrt{3}}(\tfrac{2}{3})^{\frac{4}{3}}\Gamma(\tfrac{1}{3})/\Gamma(\tfrac{2}{3}) = 2{\cdot}089.$$

Solving equation (37) for f''_0 yields

$$f''_0 = C(1 + a\Delta)^3 \quad (37')$$

where

$$C = A^{\frac{3}{2}} = 0{\cdot}342, \quad a = A^{\frac{1}{2}}B/2 = 0{\cdot}730. \quad (38)$$

If we adjust the value of c so as to be 0·332 for the case of a flat plate flow ($\Delta = 0$), values of c and a are

$$c = 0{\cdot}332, \quad a = 0{\cdot}723. \quad (38')$$

As for the temperature field, the Heaviside operator transformation corresponding to

$$\xi = \int_0^x \kappa\sqrt{(2u_{y0})}\,\mathrm{d}x$$

reduces equation (30) to

$$\frac{\partial^2[T - T_\infty]}{\partial\zeta^2} - 4\zeta s[T - T_\infty] = -[Q_t] \quad (39)$$

where

$$Q_t = \frac{2}{\kappa}\frac{T_{y0}}{u_{y0}}\frac{\partial\Psi_0}{\partial x}, \quad T_{y0} = \left(\frac{\partial T}{\partial y}\right)_0.$$

Subject to the condition that at $\eta = 0$

$$(T - T_\infty)_{\zeta\to 0} = T_0 - T_\infty + \frac{T_{y0}}{u_{y0}}\{\sqrt{(2u_{y0})}\zeta - u_0\}_{\zeta\to 0}$$

and that $T \to T_\infty$ as $\zeta \to \infty$, the solution of equation (39) gives the temperature gradient g'_0

in a similar form as for f_0'';

$$g_0' = A(Pf_0'')^{1/3}\left\{\left(-g_0 + f_0'\frac{g_0'}{f_0''}\right) + B(Pf_0'')^{1/3}\frac{g_0'}{f_0''}\Delta\right\}. \tag{40}$$

Substituting equation (37′) into (40) becomes

$$g_0' = cP^{1/3}\left\{\left(-g_0 + f_0'\frac{g_0'}{f_0''}\right) + B(cP)^{1/3}\frac{g_0'}{f_0''}\Delta\right\}(1 + a\Delta)$$

or for the further approximation

$$g_0' = -cP^{1/3}g_0\{1 + P^{1/3}f_0' + (1 + 2P^{2/3})a\Delta\}. \tag{40′}$$

Thus, we find summarily for the gas side that

$$f_{10}'' = c(1 + a\Delta_1)^3 \tag{41}$$

$$g_{10}' = -cP_1^{1/3}g_{10}\{1 + P_1^{1/3}f_{10}' + (1 + 2P_1^{2/3})a\Delta_1\} \tag{42}$$

$$w_0' = -cS^{1/3}(w_0 - w_\infty)\{1 + S^{1/3}f_{10}' + (1 + 2S^{2/3})a\Delta_1\} \tag{43}$$

where

$$\Delta_1 = f_{10} - \frac{(f_{10}')^2}{2f_{10}''}.$$

As for the liquid side, denoting

$$\hat{u} = u_0 - u_2, \quad \hat{\eta} = -\eta_2$$

and supposing that the liquid has a main velocity u_0 at infinity, we can follow the same manipulation leading to equation (37) and obtain

$$\hat{f}_0'' = A(\hat{f}_0'')^{1/3}\{\hat{f}_0' + B(\hat{f}_0'')^{1/3}\hat{\Delta}\}.$$

Since we have the relation that

$$\hat{f}_0 = -f_{20}, \quad \hat{f}_{20}' = 0, \quad \hat{f}_{20}'' = f_{20}'', \quad \hat{\Delta} \approx 0$$

the above equation yields

$$f_{20}'' = c(f_{10}')^{3/2}. \tag{44}$$

Substituting this into the boundary condition (17) gives

$$f_{10}' = \Lambda^{-2/3}(1 + a\Delta_1)^2 \approx \Lambda^{-2/3}. \tag{45}$$

With this value of f_{10}', we obtain the velocity gradient of the liquid at the interface

$$f_{20}'' = \Lambda^{-1}c(1 + a\Delta_1)^3 \approx \Lambda^{-1}c \tag{46}$$

and similarly the temperature gradient of the liquid

$$g_{20}' \approx cP_2^{1/3}g_{20}. \tag{47}$$

The temperatures g_{10} and g_{20} are determined by the conditions (13) and (18);

$$g_{10} = 1 - g_{20} = \left(-\frac{f_{10}}{c} + m_2P_2^{1/3}\right)\{m_1P_1^{1/3}(1 + \Delta_1^*) + m_2P_2^{1/3}\}^{-1} \tag{48}$$

where

$$\Delta_1^* = 1 + f_{10}'P_1^{1/3} + (1 + 2P_1^{2/3})a.$$

The normal velocity f_{10} is obtainable from the boundary condition (16) with equation (43)

$$f_{10} = \frac{2}{S}\,\frac{w_\infty - w_0}{1 - w_0}cS^{1/3}\{1 + f_{10}'S^{1/3} + (1 + 2S^{2/3})a\Delta_1\}. \tag{49}$$

When the phase-change at the interface is not so intense that $\Delta_1^2 \ll 1$, we can determine Δ_1 by equation (36), f_{10} by (49); f_{10}' by (45) and w_0 by (9) which are then

$$\Delta_1 = f_{10} - \frac{\Lambda^{-4/3}}{2c}, \quad f_{10} = 2cS^{-}\ \frac{w_\infty - w_0}{1 - w_0},$$

$$w_0 = \exp\left\{\frac{L}{R_v}\left(\frac{1}{T_b} - \frac{1}{T_{2\infty}}\right)\right\} \tag{50}$$

and finally predict f_{10}'' and g_{10}' from equations (41), (42) and (48);

$$f_{10}'' = c\left\{1 + a\left(2cS^{-2/3}\frac{w_\infty - w_0}{1 - w_0} - \frac{\Lambda^{-4/3}}{2c}\right)\right\}^3 \tag{51}$$

$$g'_{10} = -cP_1^{\frac{1}{3}}g_{10}\left\{1 + P_1^{\frac{1}{3}}\Lambda^{-\frac{2}{3}} + (1 + 2P_1^{\frac{1}{3}})a \times \left(2cS^{-\frac{2}{3}}\frac{w_\infty - w_0}{1 - w_0} - \frac{\Lambda^{-\frac{4}{3}}}{2c}\right)\right\} \quad (52)$$

where

$$g_{10} = \left(-2S^{-\frac{2}{3}}\frac{w_\infty - w_0}{1 - w_0} + m_2P_2^{\frac{1}{3}}\right)\left[m_1P_1^{\frac{1}{3}}\left\{1 + P_1^{\frac{1}{3}}\Lambda^{-\frac{2}{3}} + (1 + 2P_1^{\frac{1}{3}})a\left(2cS^{-\frac{2}{3}}\frac{w_\infty - w_0}{1 - w_0} - \frac{\Lambda^{-\frac{4}{3}}}{2c}\right)\right\} + m_2P_2^{\frac{1}{3}}\right]^{-1}.$$

The analytical results of equations (51) and (52) with (38′) are shown in Figs. 2–7 by the dotted lines. It is noted that equations (51) and (52) could be roughly valid even for larger amounts of phase-change at the interface.

Certain aspects of the results shown in Figs. 2–7 are discussed below according to these equations. Equations (41) and (42) or (51) and (52) show clearly the effect of the phase-change upon the coefficients, f''_{10} and g'_{10}. At the lower values of the latent heat of vaporization L with constant liquid temperature $T_{2\infty}(\approx T_0)$, w_0 takes larger values so that in the case of evaporation ($f_{10} < 0$) $|f_{10}|$ increases to reduce f''_{10} and $|g'_{10}|$ (see Fig. 7). With constant values of L, as T_0 approaches T_b (the boiling point), $|f_{10}|$ also increases with the decrease of f''_{10} and $|g'_{10}|$.

For condensation, ($f_{10} > 0$), the trends of f'_{10} and $|g'_{10}|$ are the reverse for evaporation. As the value of L becomes small or T_0 approaches T_b, the values of f''_{10} and $|g'_{10}|$ increase. The higher mass-fraction of vapor at infinity, w_∞ increases $f_{10}(>0)$ so as to make f''_{10} increase. Its effect upon $|g'_{10}|$ has both positive term which increases with w_∞ and the negative term of g_{10} which decreases with w_∞. Figure 5 shows that the former is predominant at the smaller values of w_∞, though the latter at the larger w_∞.

The effect of the free stream temperature of gas $T_{1\infty}$ is attributed mainly to the value of m_1, which is effective upon the value of g_{10}. In the case of condensation, its effect is thus most remarkable.

Finally, we examine the effect of the boundary condition at the interface upon the skin-friction and the heat-transfer. Figure 8 shows f''_{10} and g'_{10} for various conditions of the interface:

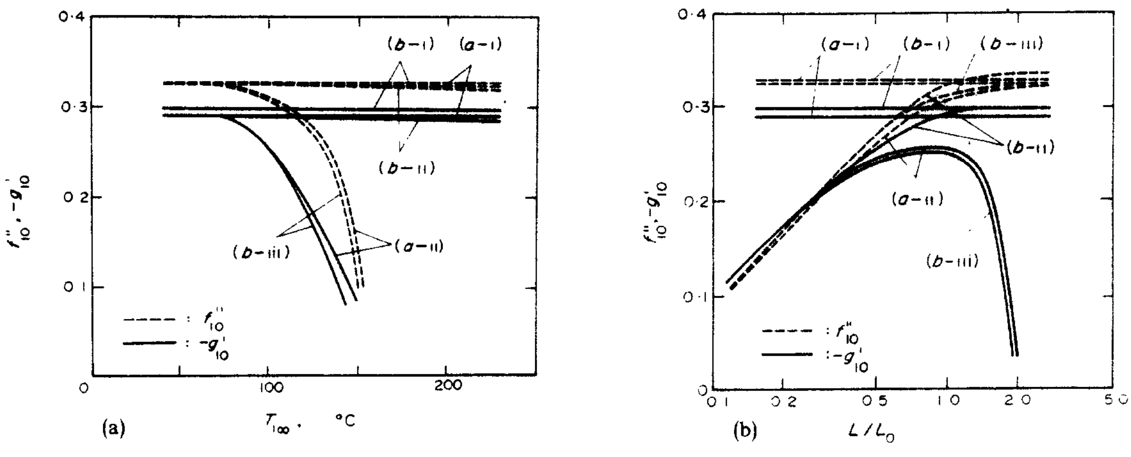

FIG. 8. Effect of the boundary condition at the interface on friction and heat-transfer coefficients (a: water, $T_{2\infty} = 20°C$, $w_\infty = 0$, $L = 539$ cal/g, b: water, $T_{1\infty} = 100°C$, $T_{2\infty} = 20°C$, $w_\infty = 0$).

(a) The liquid is unable to move, and at the interface
  (i) the phase is unchangeable, or
  (ii) the phase is to be changed.

(b) The liquid is able to move, and at the interface
  (i) the phase is unchangeable, or
  (ii) the phase is to be changed (say the normal condition), and
  (iii) in addition the temperature of the liquid is kept constant.

All of these conditions cannot always be realized physically, but are helpful to understand the basic features of the problem. The coefficients under the normal condition (b-ii) are close to those for (a-i) and (b-i) rather than for (a-ii) and (b-iii) where the phase is allowed to change, though at the lower values of latent heat they approach the values for (a-ii) and (b-iii). The conditions of (a-ii) and (b-iii) mean the heat insulation through the liquid layer ($g'_2 \equiv 0$) in which more intense evaporation ($f_{10} < 0$) takes place with the increase of $T_{1\infty}$ to make f''_{10} and $|g'_{10}|$ decrease. Under the normal condition (b-ii), the normal velocity ($-f_{10}$) at the interface is insensitive to the gas temperature, $T_{1\infty}$, though at the lower values of L it comes to be dominant likewise for (a-ii) and (b-iii).

CONCLUDING REMARKS

The features of laminar flows of gas and liquid having a phase-changing (evaporation or condensation) interface at their common boundary are investigated numerically under the assumptions of constant properties and of flat-surface boundary layers of zero pressure-gradient. The normal velocity or the value of stream function at the interface associated with the phase-changing takes an important role on the velocity profile, thus upon the coefficients of skin-friction and heat transfer. The phase-change of evaporation destabilizes the flows as evidenced by the S-shaped velocity profiles.

The analytical results (41) and (42) or (51) and (52) with an approximation for u-velocity profile, which is most closely accurate near the surface, show that the nondimensional coefficients of skin-friction and heat transfer, f''_{10} and g'_{10} are

$$\left\{1 + 0{\cdot}723\left[f_{10} - \frac{(f'_{10})^2}{2f''_{10}}\right]\right\}^3$$

and

$$\left\{1 + P_1^{\frac{1}{3}} f'_{10} + 0{\cdot}723(1 + 2P_1^{\frac{2}{3}})\left[f_{10} - \frac{(f'_{10})^2}{2f''_{10}}\right]\right\} g_{10}$$

times those for a solid flat-plate, that is 0·332, respectively, where $-f_{10}, f'_{10}, g_{10}$ are the nondimensional y- and x-components of velocity and the nondimensional temperature at the interface. These factors for the case of weakly phase-changing become

$$\left\{1 + 0{\cdot}723\left(0{\cdot}664 S^{-\frac{2}{3}} \frac{w_\infty - w_0}{1 - w_0} - 1{\cdot}51 \Lambda^{-\frac{4}{3}}\right)\right\}^3$$

and

$$\left\{1 + P_1^{\frac{1}{3}} \Lambda^{-\frac{2}{3}} + 0{\cdot}723(1 + 2P_1^{-\frac{2}{3}}) \times \left(0{\cdot}664 S^{-\frac{2}{3}} \frac{w_\infty - w_0}{1 - w_0} - 1{\cdot}51 \Lambda^{-\frac{4}{3}}\right)\right\} g_{10},$$

respectively, where w_0 and w_∞ are the mass-fractions of vapor at the interface and infinity, respectively.

TRANSFERT DE CHALEUR ET FROTTEMENT PARIETAL A UN INTERFACE AVEC CHANGEMENT DE PHASE POUR DES ECOULEMENTS LAMINAIRES GAZ–LIQUIDE

Résumé—Des écoulements laminaires de gaz et de liquide ayant à l'interface commun un changement de phase (évaporation ou condensation) sont étudiés numériquement pour des conditions de propriétés physiques constantes et pour des couches limites planes sans gradient de pression. L'accroissement de composante normale de vitesse à l'interface associée au changement de phase modifie les profils de vitesse et de température en réduisant les coefficients de frottement et de transfert thermique à l'interface. A partir d'une approximation pour le profil de vitesse, ces coefficients sont trouvés fonctions des paramètres de changement de phase à savoir les composantes normale et parallèle des vitesses et la température ou la concentration de vapeur à l'interface.

WÄRMEÜBERTRAGUNG UND REIBUNG AN DER PHASENTRENNSCHICHT EINER LAMINAREN GAS-FLÜSSIGKEITS-STRÖMUNG

Zusammenfassung—Laminare Gas- und Flüssigkeitsströmungen mit Phasenübergang an einer Trennschicht (Verdampfung oder Kondensation) als Randbedingung wurden numerisch untersucht unter Annahme konstanter Stoffwerte und ebener Grenzschichtoberflächen ohne Druckgradient. Die Zunahme der Normal-Geschwingkeit in der Trennschicht, verbunden mit dem Phansenübergang, modifiziert die Geschwindigkeits- und Temperaturprofile, so als seien die Koeffizienten für die Hautreibung und der Wärmeübergang reduziert.

Mit einer Approximation für das Geschwindigkeitsprofil können diese Koeffizenten analytisch dargestellt werden als Funktion der Phasenübergangsparameter, nämlich: Normal- und Tangentialgeschwindigkeit und der Temperatur oder der Dampfkonzentration in der Trennschicht.

ТЕПЛООБМЕН И ТРЕНИЕ НА ПОВЕРХНОСТИ РАЗДЕЛА ГАЗОЖИДКОСТНЫХ ЛАМИНАРНЫХ ТЕЧЕНИЙ ПРИ ФАЗОВЫХ ИЗМЕНЕНИЯХ

Аннотация—Проведено численное исследование газожидкостнфх потоков при фазовых изменениях (испарение или конденсация) на поверхности раздела при условии постоянных свойств и отсутствии градиента давления в плоском пограничном слое. Увеличение нормальной скорости на поверхности раздела, связанное с фазовыми превращениями, вызывает изменения профилей, скорости и температуры, приводящие к снижению коэффицентов трения и теплообмена на поверхности раздела.

При приближенном профиле скорости эти коэффициенты представлены аналитически в виде зависимостей от параметра фазлвого изменения, т.е. нормальных и параллельных скоростей и температуры или концентрации пара на поверхности раздела.